

UNCONDITIONAL BASES AND STRICTLY CONVEX DUAL RENORMINGS

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ABSTRACT. We present equivalent conditions for a space X with an unconditional basis to admit an equivalent norm with a strictly convex dual norm.

1. INTRODUCTION

We say that a norm $\|\cdot\|$ on a Banach space X is *strictly convex*, or *rotund*, if $x = y$ whenever $\|x\| = \|y\| = \frac{1}{2}\|x + y\|$. Geometrically, $\|\cdot\|$ is strictly convex if its unit sphere contains no non-trivial line segments. The norms of many classical spaces fail to possess this property, however, it is often possible to introduce a new, equivalent norm that does. Therefore we seek verifiable conditions that allow us to determine when such a *renorming* is possible.

While the notion of strict convexity has been established now for several decades, it has eluded general characterisation. In this note, we study the class of spaces X with an unconditional basis (generally uncountable). In section 2, we obtain equivalent conditions for X to admit an equivalent norm, such that its dual norm is strictly convex. The tools used are topological.

It should be noted that, in the context of spaces with unconditional bases, equivalent conditions for the existence of some other types of norm are known. We say that $\|\cdot\|$ is *locally uniformly rotund*, or *LUR*, if, given x and x_n in X such that $\|x\| = \|x_n\| = 1$ and $\|x + x_n\| \rightarrow 2$, we have $\|x - x_n\| \rightarrow 0$. Clearly, if $\|\cdot\|$ is LUR then it is also strictly convex. The norm $\|\cdot\|$ is said to be *Gâteaux smooth* if, given non-zero x , we have

$$\lim_{t \rightarrow 0} \frac{\|x + th\| + \|x - th\| - 2\|x\|}{t} = 0.$$

If, for all non-zero x , this limit exists uniformly for h in the unit sphere of X , then $\|\cdot\|$ is *Fréchet smooth*. By a well known result of Šmul'yan (cf [5, Theorem I.1.4]), if the dual norm of $\|\cdot\|$ is strictly convex (respectively LUR) then $\|\cdot\|$ is Gâteaux (respectively Fréchet) smooth. The converses do not hold, even up to renormings. In fact, there exists a space with a Fréchet smooth norm, which does not admit any equivalent norm with a strictly convex dual norm (cf [5, Theorem VII.5.2 (ii)]).

Date: November 2008.

2000 Mathematics Subject Classification. 46B03; 46B26; 46B15.

Key words and phrases. Strictly convex, norm, unconditional basis.

Supported by MTM 2008-05396/MTM/Fondos FEDER and Fundación Séneca 0690/PI/04 CARM and Institute of Mathematics and Informatics Bulgarian Academy of Sciences, and grant MM-1401/2004 of Bulgarian NSF.

However, in the class of spaces with unconditional bases, we do have equivalence up to a renorming.

Theorem 1.1. *Let X have an unconditional basis. Then X admits an equivalent norm with LUR dual norm whenever X admits an equivalent Fréchet smooth norm.*

This result has been known since the 1960s. Indeed, in [18], the second named author proved that if X has a unconditional basis then X admits an equivalent LUR norm. Since [18] is written in Russian and is of limited availability, for convenience, we define this LUR norm here. Let $\|\cdot\|$ be the original norm on X and $(e_\gamma)_{\gamma \in \Gamma}$ an unconditional basis with conjugate system $(f_\gamma)_{\gamma \in \Gamma}$. By a renorming, we may assume that $(e_\gamma)_{\gamma \in \Gamma}$ is 1-unconditional with respect to $\|\cdot\|$. Recall Day's norm $\|\cdot\|_{\text{Day}}$ on $\ell_\infty(\Gamma)$ (cf [5, Definition II.7.2]) and the fact that $\|\cdot\|_{\text{Day}}$ is LUR when restricted to $c_0(\Gamma)$ (cf [5, Theorem II.7.3]). Define $T : X \longrightarrow c_0(\Gamma)$ by $(Tx)(\gamma) = f_\gamma(x)$ for $x \in X$ and $\gamma \in \Gamma$, and set

$$|||x|||^2 = \|Tx\|_{\text{Day}}^2 + \sum_{n=0}^{\infty} 2^{-n} \|x\|_n^2$$

where

$$\|x\|_n^2 = \sup \left\{ \left\| \sum_{\gamma \in \Gamma \setminus A} f_\gamma(x) e_\gamma \right\|^2 + 2 \sum_{\gamma \in A} |f_\gamma(x)|^2 : A \subseteq \Gamma \text{ and } \text{card } A \leq n \right\}.$$

Then $|||\cdot|||$ is LUR.

Proof of Theorem 1.1. Let X have a 1-unconditional basis $(e_\gamma)_{\gamma \in \Gamma}$, with conjugate system $(f_\gamma)_{\gamma \in \Gamma}$. If X admits an equivalent Fréchet smooth norm then it cannot contain an isomorphic copy of ℓ_1 , for ℓ_1 admits no such norm (cf [5, Corollary II.3.3]). Hence $(e_\gamma)_{\gamma \in \Gamma}$ is shrinking (cf [10, Theorem 1.c.9]), and it follows that the conjugate system $(f_\gamma)_{\gamma \in \Gamma}$ is an unconditional basis of X^* . Now it is a straightforward matter to verify that the LUR norm $|||\cdot|||$ defined above, but on X^* , is w^* -lower semicontinuous and thus the dual of an equivalent norm on X . \square

In particular, if X has an unconditional basis then X admits an equivalent norm with a LUR dual norm if and only if X does not contain any isomorphic copies of ℓ_1 . In the dual strictly convex case, X may contain copies of ℓ_1 but, very roughly speaking, it cannot contain too many of them. We make this statement clearer in Remark 2.10. Note also that if Γ is uncountable and X contains an isomorphic copy of $\ell_1(\Gamma)$, then it is known that X cannot admit any equivalent norm with a strictly convex dual norm. Indeed, $\ell_1(\Gamma)$ does not even admit an equivalent Gâteaux smooth norm [4, Theorem 9].

In Section 2, we present our main topological tools, Lemma 2.1 and Theorem 2.5, together with Theorem 2.9, our characterisation. In section 3, we apply the topological tools to examples from the literature.

2. STRICTLY CONVEX DUAL NORMS

Let Γ be a set, and suppose that the real Banach space X has a normalised unconditional basis $(e_\gamma)_{\gamma \in \Gamma}$. We shall identify both X and X^* as vector sublattices of \mathbb{R}^Γ in the natural way, with the pointwise lattice order. Moreover, both sublattices are *ideals*, in the sense that if $x \in \mathbb{R}^\Gamma$, $y \in X$ and $|x| \leq y$, then $x \in X$, and similarly for X^* . By a renorming, we can assume that the basis is 1-unconditional, which means that X and X^* are both Banach lattices. It will be helpful to keep the lattice structure of X and X^* in mind. It is clear that the dual norm of X^* is always finer than $\|\cdot\|_\infty$, which we define on X^* in the obvious way.

We must define some topological concepts. A function $d : Z \times Z \longrightarrow [0, \infty)$ is called a *symmetric* if it satisfies all the axioms for a metric, with the possible exception of the triangle inequality. Let (Z, τ) be a regular topological space. We say that Z is *fragmentable* if there exists a metric d on Z with the property that, for any non-empty subset $E \subseteq Z$ and $\varepsilon > 0$, there exists a τ -open set U such that $E \cap U$ is non-empty and has d -diameter not exceeding ε . In fact, by [15, Remark 1.10], for Z to be fragmentable, we only require that d is a non-negative function on $Z \times Z$, with the property that $x = y$ whenever $d(x, y) = 0$. A family of subsets \mathcal{N} of Z is a *network* for Z if, given $x \in U \in \tau$, there exists $N \in \mathcal{N}$ such that $x \in N \subseteq U$. A family of subsets \mathcal{F} is called *isolated* if $E \cap \overline{\bigcup \mathcal{F} \setminus \{E\}}$ is empty whenever $E \in \mathcal{F}$; equivalently, there is some $U \in \tau$ such that $E \subseteq U$ and $U \cap F$ is empty for all $F \in \mathcal{F} \setminus \{E\}$. A network \mathcal{N} is called *σ -isolated* if it can be written as $\mathcal{N} = \bigcup_{n=1}^\infty \mathcal{N}_n$, where each \mathcal{N}_n is isolated. We will say that Z is *descriptive* if it is compact and admits a σ -isolated network. The class of descriptive compact spaces is broad and encompasses all metrisable compacta, as well as *Eberlein* and *Gul'ko* compacta, which we shall consider later, in Section 3. Symmetrics, σ -isolated networks and fragmentable and descriptive spaces have been studied in, for example, [7, 9, 14, 15]. We say that Z is a *Gruenhage space* if there exist families $(\mathcal{U}_n)_{n \in \mathbb{N}}$ of open sets with the property that given distinct $x, y \in Z$, there exist $n \in \mathbb{N}$ and $U \in \mathcal{U}_n$ such that $U \cap \{x, y\}$ is a singleton, and either x lies in finitely many $U' \in \mathcal{U}_n$, or y lies in finitely many $U' \in \mathcal{U}_n$. Gruenhage spaces were introduced in [8] and have recently found application in the theory of strictly convex dual norms [16]. Every descriptive compact space is Gruenhage. We let τ_p denote the pointwise topology on \mathbb{R}^Γ . We will introduce further classes of compact topological spaces in Section 3.

Lemma 2.1. *Let $K \subseteq [0, 1]^\Gamma$ be τ_p -compact satisfy $x \wedge y \in K$ whenever $x, y \in K$. Suppose further that there exists a τ_p -lower semicontinuous function $\rho : K \longrightarrow [0, 1]$ such that*

- (*) *if $y < x$ then there exists $\alpha < \rho(x)$ and an open set $U \ni y$ with the property that if $z \leq x$ and $z \in U$, then $\rho(z) < \alpha$.*

Then we deduce the following

- I. *K is fragmentable;*
- II. *for all $r \in [0, 1]$, $(\rho^{-1}(r), \tau_p)$ has a σ -isolated network.*

Moreover, if $K \subseteq \{0, 1\}^\Gamma$, then

- III. *K is descriptive.*

Proof. (I). For $x, y \in K$, set $d(x, y) = \max\{\rho(x), \rho(y)\} - \rho(x \wedge y)$. Note that (*) implies that ρ is strictly increasing, that is, $\rho(y) < \rho(x)$ whenever $y < x$. Thus, if $d(x, y) = 0$ then $x = x \wedge y = y$, so d is a symmetric. We show that it fragments K . Indeed, if $M \subseteq K$ is non-empty and $\varepsilon > 0$, let $\alpha = \sup_{x \in M} \rho(x)$. Take $x \in M$ such that $\rho(x) > \alpha - \varepsilon$. By the lower semicontinuity, there exists an open set $U \ni x$ such that $\rho(y) > \rho(x) - \varepsilon$ whenever $y \in U$. Moreover, we can assume that $y \wedge z \in U$ whenever $y, z \in U$. In particular, if $y, z \in M \cap U$ then $d(y, z) \leq \alpha - \rho(y \wedge z) < \alpha - (\alpha - 2\varepsilon) = 2\varepsilon$. By [15, Remark 1.10] mentioned in the preamble to this section, K is fragmentable.

(II). For $x \in K$ and $\varepsilon > 0$, define

$$B(x, \varepsilon) = \{y \in K : \rho(y) \leq \rho(x) \text{ and } d(x, y) < \varepsilon\}.$$

Since $y \mapsto \rho(x \wedge y)$ is τ_p -lower semicontinuous, $B(x, \varepsilon)$ is open in $\rho^{-1}[0, \rho(x)]$. We prove that if the sequence (x_n) satisfies $\max\{\rho(x), \rho(x_n)\} \rightarrow \rho(x)$ and $d(x, x_n) \rightarrow 0$, then $x_n \rightarrow x$. Indeed, first suppose that (x_{n_r}) is a subsequence converging to some $y \in K$. We have $\rho(y) \leq \liminf \rho(x_{n_r}) \leq \rho(x)$. We claim that $x \leq y$. For a contradiction, suppose otherwise. Since $x_{n_r} \rightarrow y$, we have $x \wedge x_{n_r} \rightarrow x \wedge y < x$. Thus by (*), there exists $\alpha < \rho(x)$ and an open set $U \ni x \wedge y$ such that if $z \leq x$ and $z \in U$ then $\rho(z) < \alpha$. It follows that

$$\begin{aligned} d(x, x_{n_r}) &= \max\{\rho(x), \rho(x_{n_r})\} - \rho(x \wedge x_{n_r}) \\ &\geq \rho(x) - \rho(x \wedge x_{n_r}) \\ &> \rho(x) - \alpha \\ &> 0 \end{aligned}$$

for large enough r , which is a contradiction. This proves our claim that $x \leq y$, and since $\rho(y) \leq \rho(x)$ and ρ is strictly increasing, we conclude that $y = x$. Being fragmentable and compact, K is also sequentially compact, so $x_n \rightarrow x$. Thus, if $x \in U \subseteq \rho^{-1}(r)$, with U open in $\rho^{-1}(r)$, then there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \cap \rho^{-1}(r) \subseteq U$. Consequently, d semi-metrises $\rho^{-1}(r)$ for every r . Since K is compact, it is fragmentable by a metric which generates a finer topology than τ_p [15, Corollary 1.11], and thus each $\rho^{-1}(r)$ is so fragmented. It follows that $\rho^{-1}(r)$ has a σ -isolated network by [7, Theorems 9.8 and 5.11] and [14, Lemma 2.2].

(III). Suppose now that $K \subseteq \{0, 1\}^\Gamma$. We can and do assume that $\rho(K \setminus \{\chi_\emptyset\}) \subseteq [\frac{1}{2}, 1]$. If we define

$$M = \{\lambda \chi_A : \chi_A \in K, \lambda \in [0, 2]\},$$

$\sigma(\lambda \chi_A) = \lambda \rho(A)$ for $A \neq \emptyset$, $\lambda > 0$, and $\sigma(\chi_\emptyset) = 0$, it is straightforward to verify that M is τ_p -compact and σ is a τ_p -lower semicontinuous function on M . It is clear that $x \wedge y \in M$ whenever $x, y \in M$. To see that σ satisfies (*) too, we take $x = \lambda \chi_A, y = \mu \chi_B \in M$ such that $y < x$. Clearly A is non-empty and $\lambda > 0$. In addition, we can assume that either (a) $A = B$ and $0 < \mu < \lambda$, or (b) $B \subsetneq A$ and $\mu \leq \lambda$. If (a) holds then put $\beta = \frac{1}{2}(\lambda + \mu)\rho(\chi_A) < \sigma(x)$, select $\gamma \in A$ and define the open set $V = \{z \in M : 0 < z_\gamma < \frac{1}{2}(\lambda + \mu)\}$. If $z = \nu \chi_C \leq x$ and $z \in V$ then $C \subseteq A$ and $\nu < \frac{1}{2}(\lambda + \mu)$, giving $\sigma(z) < \frac{1}{2}(\lambda + \mu)\rho(\chi_A) = \beta$. If (b) holds then take $\alpha < \rho(\chi_A)$ and an open neighbourhood $U \subseteq K$ of χ_B with the property

that $\rho(\chi_C) < \alpha$ whenever $\chi_C \in U$. Let $\beta = \frac{\lambda}{2}(\alpha + \rho(\chi_A)) < \sigma(x)$, and observe that $V = \{\nu\chi_C : \chi_C \in U, \nu < \beta/\alpha\}$ is open in M . Moreover, $y \in V$ and, if $z = \nu\chi_C \leq x$ and $z \in V$ then $\sigma(z) < \beta$.

By (II), (S, τ_p) has a σ -isolated network, where $S = \sigma^{-1}(1)$. Following [9, Theorem 7.2], (L, τ_p) also has a σ -isolated network, where $L = \sigma^{-1}[0, 1]$. We sketch an argument for completeness. The map $(t, x) \mapsto (t, tx)$ is a homeomorphism of $(0, 1] \times S$ and a subset E of $(0, 1] \times L \setminus \{0\}$, so E has a σ -isolated network. If $y = \lambda\chi_A \in L \setminus \{0\}$ then $x = \frac{1}{\sigma(y)}y = \frac{1}{\rho(\chi_A)}\chi_A$ is an element of M because $\rho(\chi_A) \geq \frac{1}{2}$. Moreover $x \in S$ and $(\sigma(y), y) \in E$. Thus E projects onto $L \setminus \{0\}$, and so $L \setminus \{0\}$ has a σ -isolated network, again by the proof of [9, Theorem 7.2]. To finish, we simply adjoin $\{0\}$ to the network. Since K embeds in L , we have proved (III). \square

Lemma 2.1 will be applied first to a specific class of topological spaces and an associated class of Banach spaces.

Definition 2.2. A family of subsets \mathcal{A} of Γ is called *adequate on Γ* , or simply *adequate*, if it satisfies the following three conditions:

- (1) $\{\gamma\} \in \mathcal{A}$ for all $\gamma \in \Gamma$;
- (2) $B \in \mathcal{A}$ whenever $A \in \mathcal{A}$ and $B \subseteq A$;
- (3) if every finite subset of $A \subseteq \Gamma$ is in \mathcal{A} then $A \in \mathcal{A}$.

Note that we can replace property (3) of Definition 2.2 with

- (3') $K_{\mathcal{A}}$ is a τ_p -compact subset of $\{0, 1\}^\Gamma$, where $K_{\mathcal{A}} = \{\chi_A : A \in \mathcal{A}\}$.

The set of totally ordered subsets of a partially ordered set E is adequate on E . If Γ is a set of consistent sentences in a first-order theory then the family of consistent subsets of Γ is adequate on Γ . Adequate families were defined in [17] and have been considered subsequently by several authors in, for example, [3, 11]. Given an adequate family \mathcal{A} , we define Banach lattice ideal $\ell_{\mathcal{A}}$ as the set of all $x \in \ell_\infty(\Gamma)$ satisfying $\|x\|_{\mathcal{A}} < \infty$, where $\|x\|_{\mathcal{A}} = \sup_{A \in \mathcal{A}} \|x \upharpoonright_A\|_1$, where $\|\cdot\|_1$ is the standard 1-norm (cf [3, Definition 2.1]). For example, if $\mathcal{A} = \{\emptyset\} \cup \{\{\gamma\} : \gamma \in \Gamma\}$ then $\ell_{\mathcal{A}} = \ell_\infty(\Gamma)$, and if $\Gamma \in \mathcal{A}$ then $\ell_{\mathcal{A}} = \ell_1(\Gamma)$. It is easy to see that, in general, the standard unit vectors $(e_\gamma)_{\gamma \in \Gamma}$ form a normalised 1-unconditional basic sequence in $\ell_{\mathcal{A}}$. We set $h_{\mathcal{A}} = \overline{\text{span}}^{\|\cdot\|_{\mathcal{A}}}(e_\gamma)_{\gamma \in \Gamma}$ and denote the dual norm on $h_{\mathcal{A}}^*$ again by $\|\cdot\|_{\mathcal{A}}$. Given $x \in h_{\mathcal{A}}$ and $A \subseteq \Gamma$, we define

$$\mathbf{1}_A(x) = \sum_{\gamma \in A} x_\gamma$$

whenever this sum makes sense. It is clear that the functions $\mathbf{1}_A$, $A \in \mathcal{A}$, lie in $h_{\mathcal{A}}^*$, with $\|\mathbf{1}_A\|_{\mathcal{A}} = 1$ whenever A is non-empty. It is also easy to verify that the map $\pi : \chi_A \mapsto \mathbf{1}_A$ on $K_{\mathcal{A}}$ is τ_p - w^* continuous; in particular, the image $\pi(K_{\mathcal{A}})$ is homeomorphic to $K_{\mathcal{A}}$. The proof of Theorem 2.5 below requires some renorming results, which we state partially.

Theorem 2.3 ([16]).

- (1) Let K be a Gruenhage compact space. Then $\mathcal{C}(K)$ admits a norm with a strictly convex dual norm.

- (2) Let (B_{X^*}, w^*) be a Gruenhage compact space. Then X admits a norm with a strictly convex dual norm.

Theorem 2.4 ([12, Theorem 2.6]). Let $(X, \|\cdot\|)$ be a Banach lattice ideal of \mathbb{R}^Γ , such that $\|\cdot\|_\infty \leq \|\cdot\|$. Then X admits a τ_p -lower semicontinuous, strictly convex norm if and only if X admits a τ_p -lower semicontinuous norm $\|\cdot\|$ satisfying $\|x\| < \|y\|$ whenever $|x| < |y|$.

Theorem 2.5. Let \mathcal{A} be an adequate family. Then the following are equivalent.

- (1) $K_{\mathcal{A}}$ is a Gruenhage compact;
- (2) $h_{\mathcal{A}}$ admits a norm with strictly convex dual norm;
- (3) there exists a strictly increasing, τ_p -lower semicontinuous map $\rho : K_{\mathcal{A}} \rightarrow [0, 1]$;
- (4) $K_{\mathcal{A}}$ is a descriptive compact.

Proof. (1) \Rightarrow (2). If \mathcal{A} is a Gruenhage compact then $\mathcal{C}(\mathcal{A})$ admits an equivalent norm $\|\cdot\|$ with a strictly convex dual norm by Theorem 2.3, part (1). Define $T : h_{\mathcal{A}} \rightarrow \mathcal{C}(\mathcal{A})$ by $(Tx)(A) = \chi_A(x)$ and observe that, since \mathcal{A} is adequate, we have $\frac{1}{2}\|x\|_{\mathcal{A}} \leq \|Tx\|_\infty \leq \|x\|_{\mathcal{A}}$. Consequently, the dual norm of $\|\cdot\|$, where $\|x\| = \|Tx\|$, $x \in h_{\mathcal{A}}$, is strictly convex. (2) \Rightarrow (3) follows from Theorem 2.4 and the natural embedding of $K_{\mathcal{A}}$ in $B_{h_{\mathcal{A}}^*}$ defined above. (3) \Rightarrow (4). If $\chi_B < \chi_A$ then take $\gamma \in A \setminus B$. Property (*) of Lemma 2.1 is fulfilled by setting $\alpha = \rho(A \setminus \{\gamma\})$ and $U = \{\chi_C : \gamma \notin C\}$. (4) \Rightarrow (1) follows from [16, Corollary 4]. \square

Remark 2.6. The proof of the implication (1) \Rightarrow (4) in Theorem 2.5 uses a renorming of a Banach space. The authors would be interested to see a direct, purely topological proof of this result. The proof of Theorem 2.5 also shows that if \mathcal{A} is adequate then $\mathcal{C}(K_{\mathcal{A}})$ admits a norm with strictly convex dual norm if and only if $K_{\mathcal{A}}$ is a Gruenhage space. We don't know if the direct implication holds in general.

We finish this section by providing a characterisation of spaces with unconditional bases which admit an equivalent norm with strictly convex dual norm. Let X have a normalised, 1-unconditional basis $(e_\gamma)_{\gamma \in \Gamma}$. Let $\|\cdot\|$ denote the dual norm on X^* . Define

$$\mathcal{A} = \{A \subseteq \Gamma : \mathbf{1}_A \in X^*\}.$$

The family \mathcal{A} contains all singletons $\{\gamma\}$, $\gamma \in \Gamma$, and is closed under taking subsets and finite unions. Hence \mathcal{A} is adequate if and only if X is isomorphic to $\ell_1(\Gamma)$. While $K_{\mathcal{A}} = \{\chi_A : A \in \mathcal{A}\}$ is not compact in general, it is σ -compact because $K_A = \bigcup_{n=1}^\infty K_{\mathcal{A}_n}$, where \mathcal{A}_n is the adequate family

$$\{A \in \mathcal{A} : \|\mathbf{1}_A\| \leq n\}.$$

Let W be the linear span of

$$\{\mathbf{1}_A : A \in \mathcal{A}, n \in \mathbb{N}\}.$$

While it is not true that $\overline{W}^{\|\cdot\|} = X^*$ in general, it is clear that $\overline{W}^{\|\cdot\|_\infty} = X^*$. We require the following result.

Proposition 2.7 ([16, Corollary 10]). *Let X be a Banach space and suppose that $X^* = \text{span}^{|||\cdot|||}(K)$, where K is a Gruenhage compact in the w^* -topology and $|||\cdot|||$ is equivalent to a coarser, w^* -lower semicontinuous norm on X^* . Then X^* admits an equivalent, strictly convex dual norm.*

A norm on a dual space X^* is said to be w^* -LUR if, given x and x_n in X^* such that $\|x\| = \|x_n\| = 1$ and $\|x + x_n\| \rightarrow 2$, we have $x_n \rightarrow x$ in the w^* topology. Such norms are studied in [14].

Theorem 2.8 ([14, Theorem 1.3]). *If B_{X^*} is a descriptive compact space in the w^* -topology then X admits an equivalent norm with w^* -LUR dual norm.*

The next result is our promised characterisation.

Theorem 2.9. *Let X have a normalised 1-unconditional basis, with \mathcal{A}_n , $n \in \mathbb{N}$, defined as above. Then the following are equivalent.*

- (1) (B_{X^*}, w^*) is a Gruenhage compact space;
- (2) X admits an equivalent norm with a strictly convex dual norm;
- (3) there exists a strictly increasing, τ_p -lower semicontinuous map

$$\rho : K_{\mathcal{A}} \longrightarrow [0, 1];$$

- (4) (B_{X^*}, w^*) is a descriptive compact space;
- (5) X admits an equivalent norm with a w^* -LUR dual norm.

Proof. (1) \Rightarrow (2) follows from Theorem 2.3, part (2). (4) \Rightarrow (5) follows from Theorem 2.8. (5) \Rightarrow (2) is an immediate consequence of the definition and (4) \Rightarrow (1) is [16, Corollary 4]. We prove (2) \Leftrightarrow (3) and (2) \Rightarrow (4). Suppose that (2) holds. Using Theorem 2.4, we can find a strict lattice dual norm $|||\cdot|||$ on X^* . It is easy to see that the map $\chi_A \mapsto |||\mathbf{1}_A|||$ satisfies (4). Now suppose that ρ satisfies (3). Let $K_n = \{\mathbf{1}_A : A \in \mathcal{A}_n\}$, where \mathcal{A}_n is as above. By Theorem 2.5, each K_n is a descriptive compact in the w^* -topology. If we set $K = \bigcup_{n=1}^{\infty} n^{-2}K_n \cup \{0\}$ then K is again descriptive, and $\overline{\text{span}}^{|||\cdot|||}(K) = \overline{W}^{|||\cdot|||} = X^*$, where W is as above. Since $|||\cdot|||$ is a w^* -lower semicontinuous norm on X^* , coarser than $\|\cdot\|$, we can apply Proposition 2.7 to obtain an equivalent, strictly convex dual norm on X^* .

To finish, we prove (2) \Rightarrow (4). Given (2), let $|||\cdot|||$ be as above, and identify the positive part B_+ of its unit ball with a τ_p -compact subset of $[0, 1]^{\Gamma}$. By applying Lemma 2.1 to B_+ with $\rho = |||\cdot|||$, we have that $\{f \in B_+ : |||f||| = 1\}$ has a σ -isolated network in the τ_p (equivalently w^*) topology. It follows from [9, Theorem 7.2] that (B_+, w^*) has a σ -isolated network, so is a descriptive compact. Because descriptive compact spaces are preserved under continuous images [13, Corollary 3.4], we have that $B \subseteq B_+ - B_+$ is descriptive. \square

Remark 2.10. We observe that $A \in \mathcal{A}$ if and only if $(x_\gamma)_{\gamma \in A} \mapsto \sum_{\gamma \in A} x_\gamma e_\gamma$ defines an isomorphism from $\ell_1(A)$ into X . Thus Theorem 2.9, part (3), is a more precise formulation of the assertion, made after the proof of Theorem 1.1, that in the dual strictly convex case X cannot contain too many copies of ℓ_1 .

Finally, we note that there is a Banach space of type $h_{\mathcal{A}}$ which does not satisfy the conditions of Theorems 2.5 or 2.9 and does not contain an isomorphic copy of $\ell_1(\Gamma)$ for any uncountable Γ . See [3, Theorems 1.7 and 3.6 (c)].

3. EXAMPLES

In this section, we apply Lemma 2.1 and Theorem 2.5 to obtain some new results concerning examples of compact spaces from the literature.

Definition 3.1. We shall say that a partially ordered set $(T, <)$ is a *pseudotree* (respectively *tree*) if, for every $x \in T$, the set $I_x = \{w \in T : w < x\}$ is totally (respectively *well*) ordered.

Pseudotrees were introduced by Kurepa and studied in [11] under the name of *bushes*. We say that a subset Γ of a partially ordered set is an *antichain* if no two distinct elements of Γ are comparable. A partially ordered set is called *special* if it can be written as a countable union of antichains. Given a pseudotree T , we let \mathcal{A}_T be the adequate family of all totally ordered subsets of T . Such families were investigated in the context of Talagrand compact spaces in [11]. A compact space K is called *Talagrand* if the Banach space $\mathcal{C}(K)$ is \mathcal{K} -analytic in its weak topology; see, for example, [1, 11, 17] for details.

Proposition 3.2 ([11, Theorem 3.2]). *Let T be a pseudotree. Then $K_{\mathcal{A}_T}$ is a Talagrand compact if and only if T is a countable union of antichains.*

We can use Theorem 2.5 to provide a straightforward extension of this result. Recall that a compact space K is called *Eberlein* if it is homeomorphic to a weakly compact subset of a Banach space. The implications Eberlein \Rightarrow Talagrand \Rightarrow descriptive \Rightarrow Gruenhage have been established and are known to be strict.

Proposition 3.3. *Let T be a pseudotree. Then the following are equivalent.*

- (1) $K_{\mathcal{A}_T}$ is Eberlein;
- (2) $K_{\mathcal{A}_T}$ is Gruenhage;
- (3) T is special.

Proof. Only (2) \Rightarrow (3) and (3) \Rightarrow (1) require proof. Assume (2). By Theorem 2.5, there exists a strictly increasing map $\rho : K_{\mathcal{A}_T} \rightarrow [0, 1]$. If we pick $\sigma(x) \in (\rho(\chi_{I_x}), \rho(\chi_{I_x \cup \{x\}})) \cap \mathbb{Q}$ for each $x \in T$, it is evident that $\sigma : T \rightarrow \mathbb{Q}$ is strictly increasing, and that the fibres $\sigma^{-1}(q)$, $q \in \mathbb{Q}$, are antichains.

Assume (3). Let $T = \bigcup_{n \in \mathbb{N}} \Gamma_n$, where Γ_n , $n \in \mathbb{N}$, is a pairwise disjoint family of antichains. It is clear that the map $\pi : K_{\mathcal{A}_T} \rightarrow c_0(T)$, defined by $\pi(\chi_A)(x) = 2^{-n}$ if $A \cap \Gamma_n = \{x\}$ for some n , and $\pi(\chi_A)(x) = 0$ otherwise, is a homeomorphism of \mathcal{A}_T and a weakly compact subset of $c_0(\Gamma)$. \square

Incidentally, using Proposition 3.3, we can provide more examples of spaces $h_{\mathcal{A}}$ which fail the conditions of Theorems 2.5 and 2.9 and do not contain isomorphic copies of $\ell_1(\Gamma)$ for any uncountable set Γ . Let T be a non-special pseudotree with no uncountable branches. Plenty of such objects exist; for example, the space $\sigma\mathbb{Q}$ of well ordered subsets of \mathbb{Q} , partially ordered by taking initial segments, satisfies

these conditions. By Proposition 3.3, $h_{K_{\mathcal{A}_T}}$ fails Theorems 2.5 and 2.9. To see that it does not contain $\ell_1(\Gamma)$ if Γ is uncountable, we refer the reader to [3, Theorem 1.7 and Proposition 3.10] and [2, Theorem 3.5].

The following class of examples is based on a compact space constructed in [3]. This construction is shown to be descriptive in [13]. Here, we give an alternative proof using our strictly increasing functions.

Example 3.4 (cf [13, Example 4.5]). Let $L = \{(\xi, \eta) \in \omega_1^2 : \xi < \eta\}$ and suppose that we have a function $\Phi : L \rightarrow \mathbb{N}$. Define

$W = \{\{\xi_1, \dots, \xi_n\} \subseteq \omega_1 : \xi_1 < \dots < \xi_n \text{ and } \Phi(\xi_i, \xi_j) \geq j \text{ for all } i < j \leq n < \omega\}$
and

$$\mathcal{A}_\Phi = \{A \subseteq \omega_1 : \text{every finite subset of } A \text{ is in } W\}.$$

Then W contains all singleton subsets of ω_1 and is closed under taking subsets, and \mathcal{A}_Φ is adequate. We show that $K_{\mathcal{A}_\Phi}$ is a descriptive compact space.

Proof. If $\chi_A \in K_{\mathcal{A}_\Phi}$ then A cannot have order type exceeding ω . Indeed, otherwise, then we could extract elements $\xi_1 < \dots < \xi_n < \dots < \xi_\omega \in A$, to give $\{\xi_1, \dots, \xi_n, \xi_\omega\} \in W$ and $\Phi(\xi_1, \xi_\omega) \geq n + 1$ for all n , which is impossible.

In order to construct a strictly increasing, τ_p -lower semicontinuous function $\rho : K_{\mathcal{A}_\Phi} \rightarrow \mathbb{R}$, we first define $\pi : K_{\mathcal{A}_\Phi} \rightarrow c_0(L)$ by

$$\pi(\chi_A)(\xi, \eta) = \begin{cases} n^{-1} & \text{if } \xi, \eta \in A \text{ and } \Phi(\xi, \eta) = n \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $\pi(\chi_A) \in c_0(L)$ because there are only finitely many $(\xi, \eta) \in L \cap A^2$ with $\Phi(\xi, \eta) = n$. Indeed, as we have already observed, we can enumerate A as a strictly increasing sequence $(\xi_i)_{i < \alpha}$, where $\alpha \leq \omega$. Thus, if $(\xi, \eta) \in L \cap A^2$ and $\Phi(\xi, \eta) = n$ then $\xi = \xi_i$ and $\eta = \xi_j$ for some $i < j$, so $\{\xi_1, \dots, \xi_j\} \in W$ and $j \leq \Phi(\xi_i, \xi_j) = n$. Evidently, if $A \subseteq B$ then $\pi(\chi_A) \leq \pi(\chi_B)$. It is also clear that if B contains at least two elements and strictly contains A , then $\pi(\chi_A)(\xi, \eta) = 0 < \pi(\chi_B)(\xi, \eta)$ for some suitable $(\xi, \eta) \in L$. Finally, we observe that π is τ_p - τ_p continuous.

We define our strictly increasing function ρ by recalling Day's norm $\|\cdot\|_{\text{Day}}$ from the introduction and setting

$$\rho(\chi_A) = \begin{cases} 0 & \text{if } A = \emptyset \\ 1 & \text{if } A \text{ is a singleton} \\ 1 + \|\pi(\chi_A)\|_{\text{Day}} & \text{otherwise.} \end{cases}$$

Since $\|\cdot\|_{\text{Day}}$ is τ_p -lower semicontinuous and lattice, ρ is τ_p -lower semicontinuous. Being strictly convex, Day's norm is moreover strictly lattice, thus ρ is strictly increasing. It follows that $K_{\mathcal{A}_\Phi}$ is descriptive by Theorem 2.5. \square

The compact space K is said to be *Gul'ko* if $\mathcal{C}(K)$ is \mathcal{K} -countably determined in its weak topology. We say that K is *Corson* if it is homeomorphic to a subset M of $[0, 1]^\Gamma$ in the pointwise topology, with the property that the support of every element of M is at most countable. See, for example, [11, 8, 2, 3, 6] for information about these classes of compact spaces. The implications Talagrand \Rightarrow Gul'ko \Rightarrow Corson and descriptive are known and strict. Since the order type of every $A \in \mathcal{A}_\Phi$ of

Example 3.4 cannot exceed ω , the associated compact space $K_{\mathcal{A}_\Phi}$ is always Corson. Using a particular function Φ defined in terms of a family of almost disjoint subsets of ω , the authors of [3] show that the associated space $K_{\mathcal{A}_\Phi}$ is not Gul'ko.

Our final collection of examples is based on a class of compact spaces introduced in [1]. We shall say that a subset I of a pseudotree T is an *interval* of T if $s \in I$ whenever $r, t \in I$ and $r \prec s \prec t$.

Example 3.5. Let A be a set and $(T_a, <_a)_{a \in A}$ a family of special pseudotrees with the property that

(**) if I is an interval of T_a and T_b for distinct $a, b \in A$, then $\text{card } I \leq 1$.

Let $T = \bigcup_{a \in A} T_a$ and define

$$\Omega = \{ \chi_I \in \{0, 1\}^T : I \text{ is an interval of } T_a \text{ for some } a \in A \}.$$

Using (**), it is straightforward to prove that Ω is τ_p -compact. We show moreover that Ω is descriptive.

Proof. Since each T_a is special, we can take a sequence $(A_{a,n})_{n=1}^\infty$ of pairwise disjoint antichains of T_a such that $T_a = \bigcup_{n=1}^\infty A_{a,n}$. If $I \in \Omega$ has at least two elements then there is a unique $a_I \in A$ such that $I \subseteq \Omega_{a_I}$. Thus we can define

$$\rho(\chi_I) = \begin{cases} 0 & \text{if } I = \emptyset \\ 1 & \text{if } I \text{ is a singleton} \\ 1 + \sum \{2^{-n} : I \cap A_{a_I,n} \neq \emptyset\} & \text{otherwise.} \end{cases}$$

It is evident that ρ is τ_p -lower semicontinuous. It remains to prove that ρ satisfies property (2.1) of Lemma 2.1. Suppose that $\chi_J < \chi_I$. If I is the singleton $\{t\}$ then set $\alpha = \frac{1}{2}$. Otherwise $I \subseteq \Omega_{a_I}$. In this case take $t \in I \setminus J$ and set $\alpha = \rho(\chi_I) - 2^{-(m+1)}$, where m is the unique natural number satisfying $t \in A_{a_I,m}$. Note that $\alpha > 1$. In both cases, define $U = \{ \chi_R \in \Omega : t \notin R \}$. It is straightforward to verify that α and U fulfil property (2.1) of Lemma 2.1. \square

The authors of [1] use families of trees $(T_a, <_a)_{a \in A}$ satisfying (**) of Example 3.5 to produce several examples of compact spaces Ω in this way, including a non-Gul'ko space. Every tree considered in [1] has height ω , so is certainly special. If each $(T_a, <_a)$ is special then none contain uncountable intervals, so the corresponding compact space Ω is Corson. From above, it follows that Ω is also descriptive. In particular, we have another example of a Corson, descriptive, non-Gul'ko space.

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